MAC-CPTM Situations Project

Situation 55: Properties of i and Complex Numbers

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Prompt

A teacher of an algebra III course notices that her students often interpret *i* in expressions like $3i$, i^2, and $4i+2$ as though it were an unknown or variable, rather than a number.

Commentary

This situation provides different interpretations of the imaginary unit and utilizes the multiplication operation involving this unit. Symbolic, graphical, and geometric representations are used in this set of foci. The first focus treats the imaginary unit as a number that is a solution to a quadratic equation with no real solutions. The second focus treats multiplication by the imaginary unit as an operation that rotates points about the origin in the complex plane. Connections are made to vector operations in the third and fourth foci, with the fifth focus incorporating linear algebra.

Mathematical Foci

Mathematical Focus 1

The imaginary number i is a solution to $x^2+1=0$ and is a special case of a complex number.

Historically, the invention of complex numbers emerged within the context of solving quadratic equations. The quadratic equation $x^2+1=0$ has no real solutions. Solving for x gives $x=\pm\sqrt{-1}$. By defining a number i, such that

$i = \sqrt{-1}$, the solutions to the quadratic equation $x^2 + 1 = 0$ can be expressed as $x = \pm i$. Substituting $x = i$ gives $i^2 = -1$.

To distinguish the number i from a variable x, consider i as a special case of a complex number. By definition, complex numbers are of the form $z = a + bi$, where a and b are real numbers. Therefore $z = i$ when $a = 0$ and $b = 1$.

Mathematical Focus 2

The multiplication operation involving -1 and i can be represented as rotations on the real number line and the complex plane.

Cases involving -1:

Representing real numbers requires only a one-dimensional system, as each real number can be represented by a unique point on a line. Multiplying a real number by -1 can be represented as the rotation of a point on the real line 180° counterclockwise about the origin to another point on the real line equidistant from the origin (i.e., a rotation without dilation).

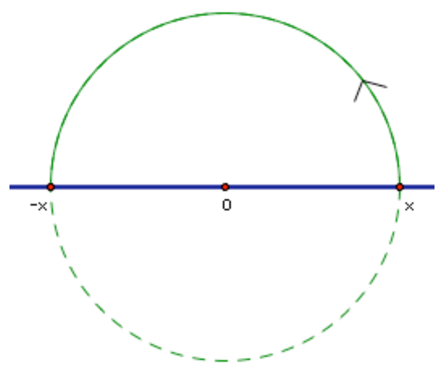

Figure 1

In contrast, representing complex numbers that have a non-zero imaginary component requires a coordinate plane, on which one axis is a real-number axis and an axis perpendicular to the first is an imaginary-number axis. In this way, a complex number, $z = x + yi$, can be represented uniquely by a point with coordinates (x, y) on the complex plane.

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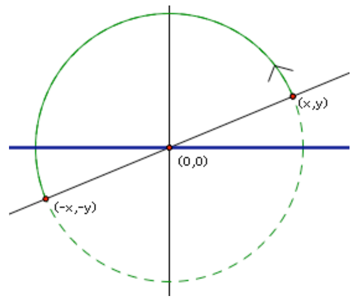

Figure 2

Cases involving i:

Since $i = \sqrt{-1}$ and $i^2 = i \cdot i = -1$, it is consistent to represent multiplying a real number by i as the rotation of a point on the real number axis 90° counterclockwise about the origin to a point equidistant from the origin on the imaginary axis.

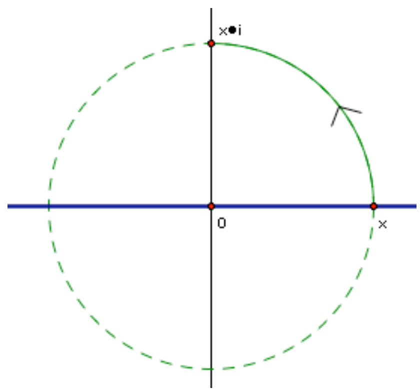

Figure 3

Multiplying a complex number $a + bi$ by i gives:
$(a + bi)i = ai + bi^2 = ai + b(-1) = -b + ai$.
Hence, multiplying a complex number by i can be represented as the rotation of a point on the complex plane 90° counterclockwise about the origin to another point on the complex plane equidistant from the origin.

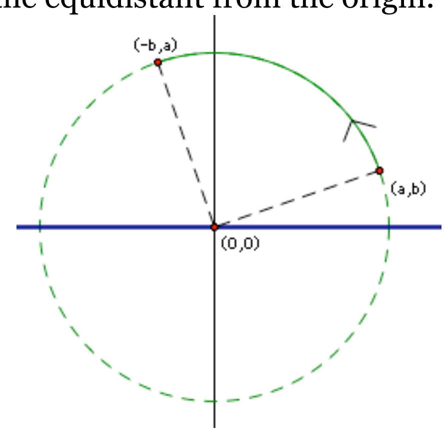

Figure 4

Mathematical Focus 3

The multiplication operation involving i can be represented as rotations of unit vectors.

Consider multiplying a positively oriented unit vector on the x-axis by i. In this way, the vector **1** is rotated 90 degrees counterclockwise about the origin to the vector ***i*** on the y-axis (figure 5).

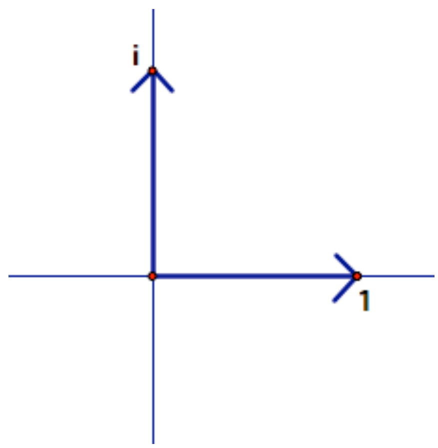

Figure 5

Now consider multiplying the vector ***i*** by i. In this way, the vector ***i*** is rotated 90 degrees counterclockwise about the origin to the vector ***i²=-1*** on the x-axis (figure 6).

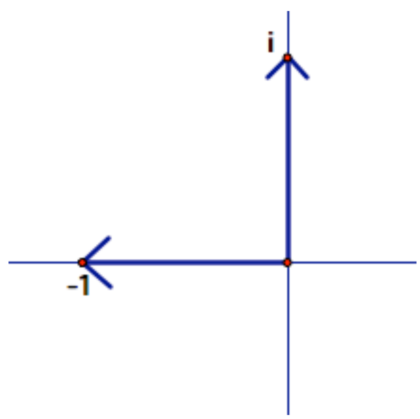

Figure 6

Mathematical Focus 4

The multiplication operation involving real and complex numbers can be represented as rotations, dilations, and linear combinations of vectors.

Now consider the multiplication of a complex number by another complex number, for example, $(2i+3)(4+i)$. The distributive property of multiplication over addition can be used to determine $(2i+3)(4+i)$ because the set of complex numbers satisfies the field properties for addition and multiplication.

The vector **(4+i)** is depicted in figure 7.

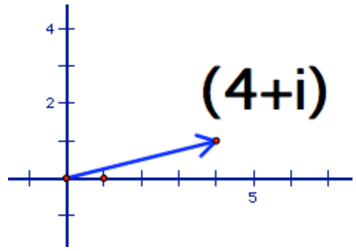

Figure 7

Determining partial products:

Multiplying the vector **(4 + i)** by 2i rotates it 90 degrees counterclockwise about the origin and dilates it by a factor of 2, as shown by the red vector in figure 8. Multiplying the vector **(4+ i)** by 3 dilates it by a factor of 3, as shown by the blue vector in figure 8.

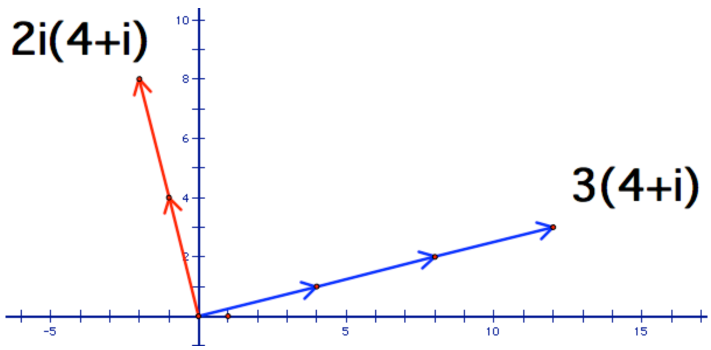

Figure 8

Summing the partial products:

The sum of the vectors **2i(4+ i)** and **3(4+ i)** is shown in figure 9.

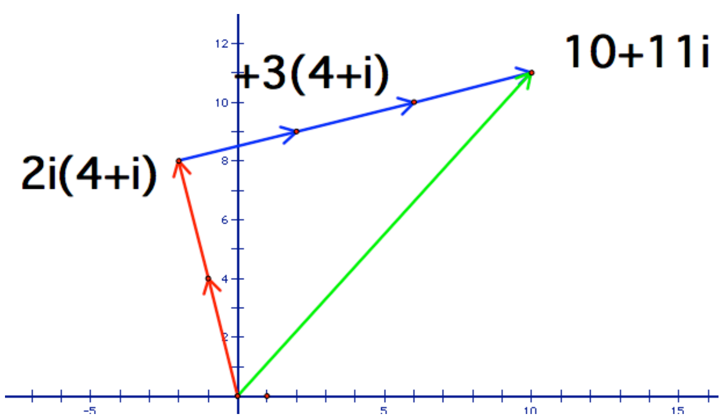

Figure 9

Hence $(2i+3)(4+i)=10+11i$.

Mathematical Focus 5

Complex numbers and their dilations and rotations can be represented with matrices.

A complex number can be represented as a matrix in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, and a rotation matrix is defined as $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ where ϑ is the angle of rotation in the counterclockwise direction. The imaginary number *bi* can be represented by a matrix of the form $\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This can be interpreted in terms of the rotation matrix as $\cos\vartheta = 0$ and $\sin\vartheta = 1$, giving $\vartheta = 90^\circ$ followed by a dilation of the scale factor *b*. In this way, multiplying by the imaginary number *bi* produces a rotation of 90° about the origin in the counterclockwise direction and a dilation of the scale factor *b*. Next, the real number *a* can be represented by a matrix of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This can be interpreted in terms of the rotation matrix as $\cos\vartheta = 1$ and $\sin\vartheta = 0$, giving $\vartheta = 0^\circ$ followed by a dilation of the scale factor *a*. In this way, multiplying by the real number *a* produces a rotation of 0° about the origin and a dilation of the scale factor *a*.